Qualifying Exam for Ph.D. Candidacy
Department of Physics
October 5th, 2019

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition)		$4.184\,\mathrm{kJ}$
1 GeV		$1.609 \times 10^{-10}\,\mathrm{J}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Indefinite integrals:

$$\int \frac{x}{(x^2+a^2)^n} dx = \frac{1}{2(1-n)} \frac{1}{(x^2+a^2)^{n-1}} n + c \quad \text{for } n \neq 0, 1 \tag{I–3}$$

I–1. A massless spring of rest length l_0 (with no tension) has a point mass m connected to one end and the other end fixed so the spring hangs in the gravity field as shown in FIGURE I–1. The motion of the system is only in one vertical plane.

i. Write down the Lagrangian.

ii. Find Lagrange's equations using variables θ, and $\lambda = (r - r_0)/r_0$, where r_0 is the rest length (hanging with mass m). Use $\omega_s^2 = k/m$, and $\omega_p^2 = g/r_0$.

iii. Assume that both λ and θ are small, and neglect all second order terms. Describe the motion that satisfies the initial conditions $\theta(t=0) = 0$, $\dot{\theta}(t=0) = \omega_p B$, $\lambda(t=0) = A$, and $\dot{\lambda}(t=0) = 0$. A and B are constants.

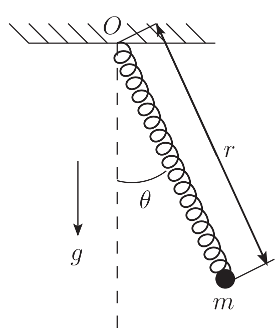

Figure I–1: Point mass hanging from a massless spring for problem I-1.

I–2. Consider a particle of mass m in the potential $V(x) = V_1(x) + V_2(x)$ with $V_1(x) = -\alpha\delta(x)$ and, for two positive constants V_0 and a,

$$V_2(x) = \begin{cases} 0 & \text{if } x < -2a \text{ or } x > -a \\ V_0 & \text{if } -2a \leq x \leq -a. \end{cases}$$

Assume that the particle has energy $E = V_0$ and that V_0 and a are such that the wave number is $ka = \frac{1}{2}\pi$.

Compute the probability that the particle is transmitted through the barrier and the probability that it is reflected by the barrier.

I–3. A) A capacitor of capacitance C and an inductor of inductance L are connected in parallel. At time zero, there is no current through the inductor and there is a charge $q(0) = Q_0$ on the capacitor. Determine the charge as a function of time $q(t)$ on the capacitor.

B) Now assume that the inductor is actually the primary coil of an ideal transformer, where the primary and secondary coil share a common magnetic flux and with symmetric primary and secondary coils (turn ratio = 1). At time zero, a resistor with resistance R is placed across the terminals of the secondary coil. Determine the differential equation for the current in the secondary as a function of time.

I–4. A quantum one-dimensional harmonic oscillator (whose ground state energy is $\hbar\omega/2$) is in thermal equilibrium with a heat bath of temperature T.

i. What is the mean value of the oscillator's energy, $\langle E \rangle$, as a function of T?

ii. What is the mean value of the root mean square fluctuations of the energy, ΔE, about the average energy $\langle E \rangle$?

iii. How do $\langle E \rangle$ and ΔE behave for $kT \ll \hbar\omega$ and $kT \gg \hbar\omega$

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II–1. Two photons, moving in opposite directions, collide along a line and can combine to produce an electron-positron pair at rest if the photon energies are just right, with each photon having energy 0.511MeV. The same resonant interaction can occur in a frame where one photon has very low energy and one much higher energy.

i. At room temperature, the black body peak wave length is about 10microns. What is the energy of such radiated photons, corresponding to this wavelength peak at room temperature.

ii. What boost Lorentz factor, $\gamma = \cosh(\text{rapidity})$, is required to change a 0.511MeV photon to one with the energy of the black body radiation emitted from room temperature (300K) objects.

iii. What photon minimum energy would begin to produce electron-positron pairs from collisions with room temperature "black body" radiated photons.

II–2. Consider a free-particle Gaussian wave packet with initial wave function at $t = 0$

$$\Psi(x, 0) = Ae^{-ax^2} e^{ilx} \ ,$$

where l is a real constant.

i. Find the normalization A and the wave function $\Psi(x, t)$ at a time t

ii. Calculate the expectation values $\langle x^2 \rangle$, $\langle x \rangle$, and uncertainty σ_x.

II–3. The volume between two concentric conducting spherical surfaces of radii a and b ($a < b$) is filled with an inhomogeneous dielectric with permittivity

$$\epsilon = \frac{\epsilon_0}{1 - Kr} \ ,$$

where K is a constant and r is radial coordinate. ($\epsilon > \epsilon_0$ for $a < r < b$). Charge Q is placed on the inner surface, while the outer surface is grounded. Find:

i. The displacement in the region $a < r < b$.
ii. The capacitance of the device.
iii. The polarization charge density in $a < r < b$.
iv. The surface polarization charge density at $r = a$ and $r = b$.

II–4. This problem refers to the radiation of black bodies.

i. Using the properties of thermodynamic potentials, derive the Maxwell relation
$$(\partial S/\partial V)_T = (\partial p/\partial T)_V$$
where S is the entropy, V the volume, T the temperature and p the pressure.

ii. From his theory of electromagnetism Maxwell found that the pressure from an isotropic radiation field and its energy desity are related as
$$p = \frac{1}{3}u(T) = \frac{U(T)}{3V}$$
where V is the volume of the cavity the radiation is enclosed in, p is its pressure, T its temperature, U is its internal energy. Using the first and second laws of thermodynamics and the result obtained at part (a) show that u obeys
$$u = \frac{1}{3}T\frac{du}{dT} - \frac{1}{3}u$$

iii. Solve the equation obtained at point (b) and obtain Stefan's law relating u and T.